

Inhomogeneous Bianchi type-I Cosmological Model with Electromagnetic Field in Lyra Geometry

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Abstract

We have investigated an inhomogeneous Bianchi type-I cosmological model with electromagnetic field based on Lyra geometry. A new class of exact solutions have been obtained by considering the potentials of metric and displacement field are functions of coordinates t and x . The physical behavior of the obtained model is discussed.

1 Introduction

After Einstein [1] proposed his theory of general relativity, which provided a geometrical description of gravitation, many physicists attempted to generalize the idea of geometrizing the gravitation to include a geometrical description of electromagnetism. One of the first attempts was made by Weyl [2] who proposed a more general theory, by formulating a new kind of gauge theory involving metric tensor to geometrize gravitation and electromagnetism. This theory was criticized due to non-integrability of length of vector under parallel displacement [3, 4].

Later Lyra [5] suggested a modification of Riemannian geometry by introducing a gauge function which removed the non-integrability condition. This modified geometry known as Lyra geometry. In a subsequent investigations Sen [6] and Sen and Dunn [4] formulated a new scalar-tensor theory of gravitation and constructed an analog of the Einstein field equations based on Lyra geometry.

For physical motivation of Lyra geometry we refer to the literature [7]-[9]. Sen [6] found that static model with finite density in Lyra geometry is similar to the static Einstein model, but a significant difference that the model exhibited red shift. Halford [10] showed in his study that the constant

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displacement vector field in Lyra geometry plays the role of cosmological constant in the theory of general relativity. Halford [11] showed that the scalar tensor treatment based in Lyra geometry predicts the same effects, within observational limits, as in Einstein theory. Several attempts have been made to cast the scalar tensor theory of gravitation in wider geometrical context [12].

Many Authors [13] have studied cosmological models in Lyra geometry with a constant gauge vector in the time-direction. The close connection between these models and general relativistic models has often been noted. Singh and his collaborators [14] have studied Bianchi type I, III, Kantowski-Sachs and new class of models with a time dependent displacement field. They have made a comparative study of Robertson-Walker models with a constant deceleration parameter in Einstein's theory with a cosmological terms and in the cosmological theory in Lyra geometry.

Bianchi-type-I cosmological model has been studied by a number of authors within the frame work of Lyra geometry. For instance, Bali and Chandnani [15] studied this model with time dependent gauge function for perfect fluid distribution. Singh and Kale [16] investigated this model with uniform and variable graviational constant with bulk viscosity. Spatially homogeneous model, which is a transform form of Bianchi type-I space-time in comoving coordinates, has been investigated, in the context of Lyra geometry, for viscous fluid distribution by Pradhan [17] and when the source of the gravitational field is perfect fluid distribution by Reddy and Venkateswarlu [18]. In this paper, we investigate the evolution of Bianchi type-I cosmological model in the presence of electromagnetic field within the frame work of Lyra geometry.

In section 2, a review for the basic equations of an anisotropic Bianchi type-I model in the presence of electromagnetic field is given. In section 3, we generate an exact solution to the Einstein field equations. Section 4, deals with the study of some physical and geometrical properties of the obtained model.

2 The metric and field equations

We consider Bianchi type-I metric in the form:

$$ds^2 = dt^2 - A^2 dx^2 - B^2 dy^2 - C^2 dz^2, \quad (2.1)$$

with the convention $(x^0 = t, x^1 = x, x^2 = y, x^3 = z)$, A is a function of t only and B and C are functions of x and t .

The volume element of the model (2.1) is given by

$$V = \sqrt{-g} = ABC \quad (2.2)$$

The four-acceleration vector, the rotation, the expansion scalar and the shear scalar characterizing the four velocity vector field, u^i , which satisfying the relation in co-moving coordinate system

$$g_{ij}u^iu^j = 1, \quad u^i = u_i = (1, 0, 0, 0),$$

respectively, have the usual definitions as given by Raychaudhuri [19]

$$\begin{aligned} \dot{u}_i &= u_{i;j}u^j, \\ \omega_{ij} &= u_{[i;j]} + \dot{u}_{[i}u_{j]}, \\ \Theta &= u^i_{;i}, \\ \sigma^2 &= \frac{1}{2}\sigma_{ij}\sigma^{ij}, \end{aligned} \quad (2.3)$$

where

$$\sigma_{ij} = u_{(i;j)} + \dot{u}_{(i}u_{j)} - \frac{1}{3}\Theta(g_{ij} + u_iu_j).$$

In view of the metric (2.1), the four-acceleration vector, the rotation, the expansion scalar and the shear scalar given by (2.3) can be written in a comoving coordinates system as

$$\begin{aligned} \dot{u}_i &= 0, \\ \omega_{ij} &= 0, \\ \Theta &= \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \\ \sigma^2 &= \frac{1}{3}\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2} - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC}\right), \end{aligned} \quad (2.4)$$

where the non vanishing components of the shear tensor σ_i^j are

$$\begin{aligned} \sigma_1^1 &= \frac{1}{3}\left(\frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \\ \sigma_2^2 &= \frac{1}{3}\left(\frac{2\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right), \\ \sigma_3^3 &= \frac{1}{3}\left(\frac{2\dot{C}}{C} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right), \\ \sigma_4^4 &= 0. \end{aligned} \quad (2.5)$$

The field equations based in Lyra geometry as obtained by Sen [6] can be written as:

$$G_{ij} + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k = -\chi T_{ij}, \quad (2.6)$$

where G_{ij} is the usual Einstein tensor, whereas ϕ_i is a displacement field vector defined by

$$\phi_i = (\beta(x, t), 0, 0, 0). \quad (2.7)$$

T_{ij} is the energy momentum tensor given by

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij} + E_{ij}. \quad (2.8)$$

where E_{ij} is the electro-magnetic field given by Lichnerowicz [20]:

$$E_{ij} = \bar{\mu}[h_l h^l (u_i u_j - \frac{1}{2} g_{ij}) + h_i h_j] \quad (2.9)$$

Here ρ and p are the energy density and isotropic pressure respectively and $\bar{\mu}$ is the magnetic permeability and h_i the magnetic flux vector defined by:

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}} \epsilon_{ijkl} F^{kl} u^j \quad (2.10)$$

F_{ij} is the electromagnetic field tensor and ϵ_{ijkl} is the Levi-Civita tensor density. If we consider that the current flow along z-axis, then F_{12} is only non-vanishing component of F_{ij} .

The Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \quad (2.11)$$

and

$$[\frac{1}{\bar{\mu}} F^{ij}]_{;j} = J^i \quad (2.12)$$

require that F_{12} be function of x alone [21]. We assume that the magnetic permeability as a function of x and t both. Here the semicolon represents a covariant differentiation.

For the line element (2.1) the field equation (2.6) with equation (2.8) lead to the following system of equations

$$\frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{1}{A^2} \frac{B'C'}{BC} + \frac{3}{4}\beta^2 = -\chi(p - \frac{F_{12}^2}{2\bar{\mu}A^2B^2}), \quad (2.13)$$

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{C}}{C} - \frac{1}{A^2} \frac{C''}{C} + \frac{3}{4}\beta^2 = -\chi(p - \frac{F_{12}^2}{2\bar{\mu}A^2B^2}), \quad (2.14)$$

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{B}}{B} - \frac{1}{A^2} \frac{B''}{B} + \frac{3}{4}\beta^2 = -\chi(p + \frac{F_{12}^2}{2\bar{\mu}A^2B^2}), \quad (2.15)$$

$$\frac{\dot{B}'}{B} - \frac{\dot{A}B'}{AB} + \frac{\dot{C}'}{C} - \frac{\dot{A}C'}{AC} = 0, \quad (2.16)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{1}{A^2} \left(\frac{B''}{B} + \frac{C''}{C} + \frac{C'B'}{CB} \right) - \frac{3}{4}\beta^2 = \chi \left(\rho - \frac{F_{12}^2}{2\bar{\mu}A^2B^2} \right), \quad (2.17)$$

where the over heat dot denotes differentiation with respect to t and over head prime denotes differentiation with respect to x .

3 Solutions of the Field Equations

The field equations (2.13)-(2.17) constitute a system of five highly non-linear differential equations with seven unknowns variables, A, B, C, p, ρ, F_{12} and β . Therefore, two physically reasonable conditions amongst these parameters are required to obtain explicit solutions of the field equations.

First, let us assume that the expansion scalar Θ in the model (2.1) is proportional to the eigenvalue σ_1^1 of the shear tensor σ_i^j , then from (2.4) and (2.5) we get,

$$\frac{1}{3} \left(\frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) = \alpha \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \quad (3.1)$$

where α is a constant of proportionality, hence;

$$\frac{\dot{A}}{A} = \left(\frac{3\alpha + 1}{3} \right) \frac{\dot{A}BC + \dot{B}AC + \dot{C}AB}{ABC}, \quad (3.2)$$

by integration we have:

$$A = (BC)^n, \quad (3.3)$$

where $n = \frac{3\alpha+1}{2-3\alpha}$ is a constant.

The motive behind assuming this condition is explained as follow [22]: Referring to Thorne [23], the observations of velocity-redshift relation for extra-galactic sources suggest that the Hubble expansion of the universe is isotropic to within ≈ 30 per cent [24, 25]. More precisely, the redshift studies place the limit $\frac{\sigma}{H} \leq 0.30$ where σ is the shear and H the Hubble constant. Collins et al. [26] have pointed out that for a spatially homogeneous metric, the normal congruence to the homogenous hypersurface satisfy the condition $\frac{\sigma}{\Theta} = \text{constant}$.

The second required condition is by assuming that the density ρ and the pressure p are related by barotropic equation of state

$$p = \lambda \rho \quad 0 \leq \lambda \leq 1. \quad (3.4)$$

From the condition (3.3) we can make the following assumptions

$$B(x, t) = f(x)k(t), \quad C(x, t) = \frac{l(t)}{f(x)}. \quad (3.5)$$

Note that other choices can be made and thus the above definitions in equation (3.5) are by no means unique.

If we substitute the assumptions in equation (3.5) - take into account the condition (3.3) - into equation (2.16), we obtain the following relation for the functions $k(t)$ and $l(t)$ as follows

$$k = c_1 l, \quad (3.6)$$

where c_1 is a constant of integration.

Using (3.3), (3.5) and (3.6) in equations (2.13) and (2.14), we get the following two equations

$$\frac{3f'^2}{f^2} - \frac{f''}{f} = a, \quad (3.7)$$

where a is a constant,

$$\frac{\ddot{l}}{l} + (2n+1)\frac{\dot{l}^2}{l^2} = ml^{-4n}, \quad (3.8)$$

where $m = \frac{a}{(2n-1)c_1^{2n}}$ is a constant.

The solutions of the above two equations (3.7) and (3.8) are as follows

$$l(t) = (c_3 t + c_4)^{\frac{1}{2n}}, \quad (3.9)$$

$$f(x) = (c_5 e^{bx} + c_6 e^{-bx})^{\frac{-1}{2}}, \quad (3.10)$$

where $c_3 = n\sqrt{2m}$, $c_4 = 2n$, $b = \sqrt{2a}$, c_5 and c_6 are constants. Consequently, equation (3.6) becomes

$$k(t) = c_1 (c_3 t + c_4)^{\frac{1}{2n}}. \quad (3.11)$$

It is observed from equations (3.3), (3.5) and (3.9)-(3.11) the metric potentials A , B and C can be singular only for $t \rightarrow \infty$. Thus the line element

with these coefficients is singular free even at $t = 0$, and can be written in the following form

$$ds^2 = dt^2 - c_1^{2n}[c_3t + c_4]^2 dx^2 - c_1^2[c_3t + c_4]^{\frac{1}{n}}(c_5e^{bx} + c_6e^{-bx})^{-1} dy^2 - [c_3t + c_4]^{\frac{1}{n}}(c_5e^{bx} + c_6e^{-bx}) dz^2. \quad (3.12)$$

The above line element can be transformed, by a proper choice of constants, to the following line element

$$ds^2 = dt^2 - a_1^2(t) dx^2 - a_2^2(t) e^{2qx} dy^2 - a_3^2(t) e^{-2qx} dz^2, \quad (3.13)$$

where $q = \frac{b}{2}$. The line element (3.13) represents a Bianchi type- VI_0 .

4 Physical Properties of the Model

Using equations (3.5) and (3.9)-(3.11) in equations (2.13)-(2.17), take into account the condition (3.3), the expressions for density ρ , pressure p , electromagnetic field F_{12} and displacement field β are given by

$$\rho = \frac{c_3^2}{\chi(1-\lambda)n^2} T^{-2}, \quad (4.14)$$

$$p = \frac{\lambda c_3^2}{\chi(1-\lambda)n^2} T^{-2}, \quad (4.15)$$

$$F_{12}^2 = \frac{\bar{\mu} c_1^{2(1+n)}}{\chi} \psi^{-1} T^{\frac{1}{n}} \left[\frac{(1-2n)c_3^2}{2n^2} + \frac{b^2}{c_1^{2n}} \left(\frac{\psi_o^2}{\psi^2} - \frac{1}{2} \right) \right], \quad (4.16)$$

$$\frac{3}{2} \beta^2 = \left[\frac{2\lambda c_3^2}{(\lambda-1)n^2} - 2c_3^2 - \frac{(1-4n^2)c_3^2}{2n^2} + \frac{b^2}{2c_1^{2n}} \frac{\psi_o^2}{\psi^2} \right] T^{-2}, \quad (4.17)$$

where

$$\psi_o = c_5 e^{bx} - c_6 e^{-bx}, \quad \psi = c_5 e^{bx} + c_6 e^{-bx}$$

and

$$T = c_3 t + c_4$$

For the line element (3.12), using equations (2.2), (2.4) and (2.5), we have the following physical properties:

The volume element is

$$V = c_1^{n+1} T^{\frac{n+1}{n}}. \quad (4.18)$$

This equation shows that the volume increases as the time increases, that is, the model (3.12) is expanding with time.

The expansion scalar, which determines the volume behavior of the fluid, is given by:

$$\Theta = \frac{c_3(n+1)}{nT} \quad (4.19)$$

The non-vanishing components of the shear tensor, σ_i^j , are

$$\sigma_1^1 = \frac{(2n-1)c_3}{3nT}, \quad (4.20)$$

$$\sigma_2^2 = \frac{(1-2n)c_3}{6nT}, \quad (4.21)$$

$$\sigma_3^3 = \frac{(1-2n)c_3}{6nT}, \quad (4.22)$$

$$\sigma_4^4 = 0. \quad (4.23)$$

Hence, the shear scalar σ , is given by

$$\sigma = \frac{c_3(2n-1)}{2\sqrt{3}nT} \quad (4.24)$$

Since $\lim_{t \rightarrow \infty}(\frac{\sigma}{\Theta}) \neq 0$, then the model (3.12) does not approach isotropy for large value of t . Also the model does not admit acceleration and rotation, since $\dot{u}_i = 0$ and $\omega_{ij} = 0$. We can see that

$$\frac{\sigma_1^1}{\Theta} = \frac{2n-1}{3(n+1)} = \alpha, \quad (4.25)$$

which is a constant of proportional.

5 Discussion and Conclusion

We have offered an investigation of an anisotropic and an inhomogeneous cosmological model of Bianchi type-I with electromagnetic field in the context of Lyra geometry. We got an interesting model which represents Bianchi type- VI_0 . This model represents shearing and non-rotating. Moreover, this model is singular free even at the initial epoch $t = 0$ and has vanishing accelerations. We found also that $\lim_{t \rightarrow \infty}(\frac{\sigma}{\Theta}) \neq 0$, this means that the model does not approach isotropy for large time t .

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